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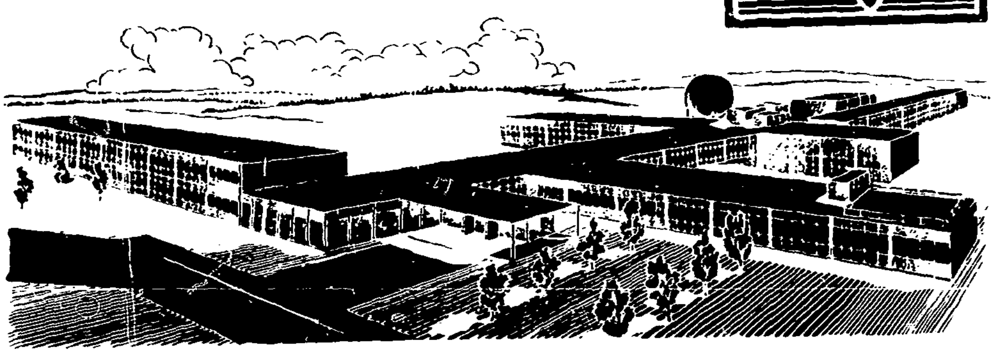
STRONG INTERACTION WITH SLIP BOUNDARY CONDITIONS

J. AROESTY

UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA

SEPTEMBER 1961

AERONAUTICAL RESEARCH LABORATORY
OFFICE OF AEROSPACE RESEARCH
UNITED STATES AIR FORCE



University of California, Berkeley, Calif. STRONG INTERACTION WITH SLIP BOUNDARY CONDITIONS by J. Aroesty, September 1961. 35 p. incl illus. (Project 7064; Task 70169) ARL 64

Unclassified Report

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AERONAUTICAL RESEARCH LABORATORY
OFFICE OF AEROSPACE RESEARCH
UNITED STATES AIR FORCE
WRIGHT-PATTERSON AIR FORCE BASE, OHIO

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FOREWORD

This report was prepared at the Institute of Engineering Research, University of California, Berkeley, California, under Contract AF 33(616)-6161, Project 7064, "Research on Aerodynamic Flow Fields," Task 70169, "Thermo-Aerodynamic Characteristics at Hypersonic Mach Numbers." The work was administered under the direction of the Aeronautical Research Laboratory, Office of Aerospace Research, USAF, with Lt John Anderson as Task Scientist.

The theoretical analysis and experimental work were carried out by J. Aroesty under the supervision of Professor G. J. Maslach and Professor S. A. Schaaf.

ABSTRACT

A solution to the problem of strong interaction between the shock wave and the boundary layer has been obtained for the case where velocity slip and temperature jump boundary conditions are consistent at the wall. It is shown that the addition of slip boundary conditions yields a correction of order $[\delta/\chi]$ to the no slip solution.

Estimates are made of the effect of slip on induced pressures and skin friction for the case of the adiabatic wall. In addition, it is shown that the inclusion of slip boundary conditions does not change the energy transfer to the wall from the no slip values.

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LIST OF SYMBOLS

M	Mach Number
C	Chapman Rubesin Constant $= \frac{\mu_w T_\infty}{\mu_\infty T_w}$
$\bar{\chi}$	Hypersonic Interaction Parameter $= \frac{M^3\sqrt{C}}{\sqrt{Re_y}}$
Re_x	Reynolds Number $= \frac{U_\infty x}{\nu_\infty}$
T	Temperature
p	Pressure
γ	c_p/c_v
δ	Boundary Layer Thickness
u	Velocity in x direction
v	Velocity in y direction
ψ	Stream Function
ξ	Transformed Variable Defined by $\xi = \int_0^x \rho_b \mu_b u_\delta \, dx$
η	Transformed Variable Defined by $\eta = \frac{u_\delta}{\sqrt{2\xi}} \int \rho \, dy$
f	Defined by $\psi = \sqrt{2\xi} f$ in Boundary Layer
N	$\rho\mu/\rho_w\mu_w$
τ	Shear Stress
λ	Local Mean Free Path
ε	Expansion Parameter $= \frac{\lambda_w \rho_w u_\delta}{\sqrt{2\xi}}$
β	Pressure Gradient Parameter, $\beta = \frac{2 \, du_\delta}{u_\delta} \frac{\xi}{d\xi} \frac{H_\delta}{h_\delta}$
h	Enthalpy
H	$h + \frac{1}{2}u^2$
p	Pressure
g	H/H_δ
δ^*	Displacement Thickness

$\overline{C}_F$ Skin Friction Coefficient $\tau / \frac{1}{2}\rho_\infty U_\infty^2$

C_{F_0} $\overline{C}_{F_0} \sqrt{\frac{Re_x}{C}} \cdot \frac{1}{\chi^{1/2}}$

Θ Momentum Thickness

Subscripts

e Edge of Boundary Layer

b Wall or Body

w Wall or Body

∞ Free Stream Values

0 Zeroth Order quantities

1 First Order Perturbation

STRONG INTERACTION WITH SLIP BOUNDARY CONDITIONS

J. Aroesty

1 INTRODUCTION

The status of the hypersonic leading edge problem has not changed appreciably in the period since the earliest recognition of the shock wave-boundary layer interaction phenomenon. The theoretical aspects of the problem have mainly been considered from the viewpoint of boundary layer theory, with few exceptions[1,2,3]. There has been some work by Charwat[4] on the application of a near-free-molecule technique to the region very close to the tip. This approach, which actually considers the various types of collisions encountered by molecules which have been reflected off the plate surface, is similar to the "physical iteration" scheme used by Charwat and Baker[5] in their consideration of near-free-molecule-flow over a sphere. This counting of collisions becomes quite complicated as more and more collisions occur and it becomes increasingly difficult to assign a mean free path to each type of molecular encounter.

Another attack on the problem, not within the scope of boundary layer theory, was carried out by Laurmann[6], who considered the problem of slip flow over a short flat plate. The solution was based on a linearization of the full Navier-Stokes equations about the free stream velocity. The interesting possibility was raised that for this problem, with sufficient slip at the surface, the Oseen solution could be valid over the entire short plate. Although it is difficult to relate this problem of the short plate to that of the leading edge region of a long flat plate on physical grounds, Laurmann's[6] result of a near linear displacement thickness is certainly suggestive of the possible effects of slip boundary conditions at the surface.

Except for a recent paper by Oguchi[7], the remainder of the analytical attempts using the continuum approach have been within the framework of hypersonic similar solutions of the boundary layer equations. The earliest solutions[8] suggested the possibility of a strong-interaction region, where the pressure grows like $x^{-1/2}$ and the boundary layer grows like $x^{3/4}$. The important parameter in this region was shown to be $\frac{M^3\sqrt{C}}{\sqrt{R_x}}$, and all solutions were given as functions of this hypersonic interaction parameter, $\bar{\chi}$.

Although the early measurements of the pressure distributions on the sharp leading edge flat plate in hypersonic flow seemed to confirm the existence of this strong interaction region, later work done at Berkeley and at Schenectady demonstrated clearly that this parameter alone is certainly not able to describe the pressures as the plate tip is approached. Starting from the downstream side, it becomes apparent that the pressures increase, at least qualitatively, as predicted by the strong interaction results until they reach a plateau of constant pressure, and then decrease. The level of the pressure plateau is a function of Mach number for sufficiently large values of $\bar{\chi}$. Since the various data have been taken at a variety of wall temperatures, it is simplest to characterize the experiments done at Berkeley as the insulated case, and the experiments done at General Electric Company as the highly cooled case.[9,10]

Even the strong interaction theories suggest that the induced pressures increase with increased T_w/T_o, so that it is not surprising that the level of this pressure plateau is also a function of this temperature ratio. Examination of the available data suggests the possibility that the extent of the strong interaction region, where the pressure grows like $x^{-1/2}$, is probably quite small, if it exists at all.

Prior to the publication of the data which showed the existence of a pressure plateau near the tip, attempts to modify the early strong interaction

solutions were made by Oguchi and Stewartson[11,12], using hypersonic small disturbance theory. On purely analytical grounds, these more recent solutions are attractive in that they provide a complete solution to the flow field between the shock wave and the boundary layer, whereas the early work, which utilized the tangent wedge formula, gave no information at all concerning the nature of the flow field external to the boundary layer.

2 OGUCHI'S ANALYSES

Recently Oguchi[11] gave a more complete joining of the boundary layer and the inviscid flow, where the vorticity and temperature at the edge of the boundary layer are no longer taken to be zero, as in the earlier solutions, but rather at the values that obtain from the solution of the zero order inviscid equations. Thus, the complete similarity of the flow field enables one to obtain information which is not at all available in the tangent wedge approach, which yields p/p_∞ alone at the edge of the boundary layer.

However, the means necessary to achieve complete similarity is to assume that the shock wave grows like Ax^n, where $1 > n \geq 2/3$. With the application of the approximate conservation laws across the shock, this implies that the pressure and temperature approach a singularity as the leading edge is approached. Thus, this simple form of the shock shape, essential to a similar inviscid flow field, is then responsible for the high pressures near the leading edge, and also for the perhaps spuriously hot gas at the edge of the boundary layer.

The matching between the shock layer and the boundary layer problem is effected by an expansion in a small parameter, $\left(\frac{M}{\sqrt{Re_\infty}}\right)^{1-2/3\gamma}$, which is suggested by the forms of the temperature and vorticity variation at the

edge of the viscous regions. For air, the small parameter is effectively of order $\frac{\delta}{x}$, which introduces some question concerning its validity in a boundary layer solution. The additional consideration of the vorticity due to the curved leading edge shock wave yields pressure levels which are higher even than those of the earlier solutions.

In the light of experiments that seem to point to a pressure plateau as the leading edge is approached, it is possible that the leading edge shock wave is not very highly curved, and in fact, it is certainly possible that it is nearly straight. With this in mind, Oguchi[7] has recently attempted to settle this problem of shock-boundary layer interaction in a manner which is similar to the viscous layer problem on a blunt body. Oguchi has assumed that the flow in the entire region between the shock wave and the body may be represented by the boundary layer equations. This implies that the entire pressure field in the very leading edge region is constant, since the assumption is also made that the shock wave is straight. In addition to the usual condition that $u = u_\infty$ at the shock, it is also required that the mass flow entering the shock wave is equal to the mass flow in the boundary layer. This is done by equating the boundary layer stream function at the still unknown shock wave height, y_s, to the stream function corresponding to the free stream at that height.

Using the nomenclature of reference [8], this new condition is

$$\psi_{bl} = \psi_{FREE\ STREAM} \tag{1}$$

$$\sqrt{2\xi}\, f(\eta_s) = \rho_\infty u_\infty y_s \tag{2}$$

where

$$\eta_s = \frac{u_\infty}{\sqrt{2\xi}} \int_0^{y_s} \rho \, dy \tag{3}$$

It is Oguchi's hope that the solution of the boundary layer equations, with this additional boundary condition, yield a zero value for the shear stress just behind the shock wave. This is critical to the justification of the use of the boundary layer equations in the entire field, since the shock wave is assumed to be a straight discontinuity, satisfying the appropriate Rankine-Hugoniot conditions. Since the shock thickness varies inversely as the strength, it is more likely that one is able to give an a priori justification for the neglect of the shock structure in the case of a very strong shock, such as a normal shock, than for this case of a comparatively weak shock.

In attempting to meet the boundary conditions for this problem, Oguchi utilizes an assumed form for the stream function which is inconsistent with the resulting equations, except for the region where $\eta_s \ll 1$, corresponding to the region closest to the tip. For larger values of η_s, a numerical solution is proposed which is called a "local similarity solution". By means of these two solutions, one analytic and one numerical, the induced pressures over the entire leading edge region are predicted, and are shown to agree quite well with the more recent experiments.

However, if the value for the shear behind the edge of the shock is evaluated properly, i.e.,

$$\tau = \frac{N \rho_w \mu_w U_\infty^2 f_{\eta\eta}(\eta_s)}{\sqrt{2\xi}} \tag{4}$$

or

$$\frac{\tau_{edge}}{\tau_w} = \frac{N f_{\eta\eta}(\eta_s)}{f_{\eta\eta}(0)} \tag{5}$$

,

the solution for the limiting case of $\eta_s << 1$ yields the result that $\frac{\tau}{\tau_w} = 1$ throughout the boundary layer. Similarly, for the case corresponding to the "locally similar" solutions, $\frac{T_{edge}}{T_{WALL}}$ is always greater than zero, and in fact, is always greater than 0.3. Thus, we see that the requirement of the vanishing of the shear behind the shock wave is not met, and an inconsistency develops at the shock wave. The way out of this inconsistency is certainly not clear, but it is apparent that the solution, as it stands now, is certainly open to question.

3 EFFECTS OF SLIP

Laurmann's suggestion that a linear displacement thickness is possible for large slip velocities, and the well known incompressible result, that small slip at the surface decreases the displacement thickness, all point to the likelihood that inclusion of first order slip boundary conditions in the strong interaction problem will tend to reduce the curvature of the boundary layer thickness, and consequently reduce the pressure levels from the no slip value as the leading edge is approached. Street[13] and Laurmann[14] have considered this problem, the latter using an integral technique, and the former using an expansion procedure somewhat analogous to the Oguchi procedure for matching the vorticity.

The procedure which Street and Laurmann used is the expansion of certain flow quantities in terms of a small parameter, which is related to the local mean free path at the wall. This technique is quite well known[15], and is characterized as possessing solutions which are related to the y-derivatives of the zeroth order solutions.

Cassaccio[15] has attempted to generalize this result to the case of a similar compressible flow, where no effect on the external flow is considered.

The solutions obtained are characterized by this linear dependence on the y-derivatives of the zeroth order solution.

If the expansion is cast in the Howarth-Dorodnitsyn variables[8], the appropriate expansion parameter becomes evident from the expression for $\frac{\partial u}{\partial y}$ at the wall, i.e., slip boundary conditions.

$$u(0) = a\lambda \frac{\partial u}{\partial y} \qquad (6)$$

$$\sim u(0) = \frac{\lambda_w \rho_w u_s}{\sqrt{2\xi}} f_{\eta\eta} \cdot a \qquad (7)$$

The appropriate expansion parameter becomes

$$\varepsilon = \frac{\lambda_w \rho_w u_s}{\sqrt{2\xi}} \qquad (8)$$

Expanding $$\frac{u}{u_s} = f_0(\eta) + \varepsilon f_1(\eta) + o(\varepsilon^2) + \cdots \qquad (9)$$

(i.e., perturbing a similar solution), and inclusion in the momentum equation yields

$$N f_{0\eta\eta\eta} + f_0 f_{0\eta\eta} + \beta(g_0 - f_{0\eta}^2) = 0 \qquad \text{zero}^{th} \text{ order in } \varepsilon \qquad (10)$$

$$\varepsilon N f_{1\eta\eta\eta} + \varepsilon f_0 f_{1\eta\eta} + \varepsilon f_1 f_{0\eta\eta} + \varepsilon\beta g_1 - 2\varepsilon\beta f_{0\eta} f_{1\eta} \qquad \text{first order in } \varepsilon$$
$$= 2\xi \frac{d\varepsilon}{d\xi}\left(f_{0\eta} f_{1\eta} - f_1 f_{0\eta\eta}\right) \qquad (11)$$

The requirement for similarity in the zeroth order solution was that

$$\beta = 2\frac{du_\delta}{u_\delta}\frac{\xi}{d\xi}\,\frac{H_\delta}{h_r} \tag{12}$$

or
$$\frac{u_\delta^2}{H_\delta - \frac{1}{2}u_\delta^2} = B\xi^\beta \tag{13}$$

For the assumed first order expansion to be consistent, it is clearly necessary that ε must be of the form $B\xi^K$ when K is a constant. Since $\lambda_w \rho_w$ is independent of ξ, being determined only by the wall temperature, this is equivalent to requiring that u_δ vary like a power of ξ. This is inconsistent with the requirement for the zeroth order solution, which is that

$$u_\delta = \left\{\frac{B\xi^\beta}{1-\frac{1}{2}\frac{B\xi^\beta}{H_\delta}}\right\}^{1/2} \tag{14}$$

Only for the special cases of $\beta = 0$, corresponding to a flat plate, $M_\delta^2 \gg 1$, corresponding to hypersonic flow, or $M_\delta^2 \ll 1$ corresponding to low speed flow, is the expansion procedure completely consistent. Maslen[17] gave the complete solution for the case where $\beta = 0$, corresponding to compressible slip flow over a flat plate. Otherwise, it is possible to show that the only solution satisfying this assumed form for f is the one which drives the term in the brackets in equation (11) identially to 0, or $f_1 = A f_\eta$. However, this is certainly a very special case, since we have only given one solution to a third order differential equation for f_1. In general, we can see that this assumed expansion is not satisfactory.

4 PRESENT ANALYSIS

The present analysis will consist of a perturbation about the strong interaction solution, in order to represent the effects of the slip boundary conditions.

The following assumptions will be made in order to obtain the familiar zero order equations given in [8], page 305.

1. The boundary layer equations are valid.
2. Pr = 1
3. $\frac{\rho\mu}{\rho_b\mu_b} = N =$ (constant)
4. $M_\delta >> 1$, H_δ = constant
5. Tangent wedge formula is valid to describe the interaction between the displacement thickness and the induced pressure.

The boundary layer equations, with first order slip, for Pr = 1 may be written as:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \qquad (15)$$

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \qquad (16)$$

$$\rho u \frac{\partial H}{\partial x} + \rho v \frac{\partial H}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial H}{\partial y}\right) \qquad (17)$$

B.C. $y = 0,\ v = 0,\ u(0) = a\lambda \left.\frac{\partial u}{\partial y}\right|_0$ (18) λ = local mean free path

$$T(0) = T_w + b\lambda \left.\frac{\partial T}{\partial y}\right|_0 \qquad (19)$$

$$y = \delta,\ u = u_\delta,\ H = H_\delta \qquad (20)$$

4.1 Derivation of the Zero & First Order Equations

The Howarth Dorodnytsin transformations are applied in order to render the equations amenable to a similar solutions analysis.

$$\xi = \int \rho_b \mu_b u_\delta \, dx \qquad (21)$$

$$\eta = \frac{u_\delta}{\sqrt{2\xi}} \int \rho \, dy \qquad (22)$$

$$g = H/H_\delta \qquad (23)$$

$$\psi = \sqrt{2\xi}\, f(\eta, \xi) \quad \text{where} \quad f_\eta = \frac{u}{u_\delta} \qquad (24)$$

The momentum equation becomes

$$N f_{\eta\eta\eta} + f f_{\eta\eta} + 2\frac{d\ln u_\delta}{d\ln \xi}\left(\frac{\rho_\delta}{\rho} - f_\eta^{\,2}\right) = 2\xi\left(f_\eta f_{\eta\xi} - f_\xi f_{\eta\eta}\right) \qquad (25)$$

The energy equation becomes

$$N g_{\eta\eta} + f g_\eta = 2\xi\left(f_\eta g_\xi - f_\xi g_\eta\right) \qquad (26)$$

The momentum equation can be put in a somewhat different form for the case where the external flow is hypersonic-- i.e., $M_\delta^{\,2} \gg 1$.

$$\frac{d\ln u_\delta}{d\ln \xi} = \frac{\xi}{d\xi}\,\frac{du_\delta}{u_\delta} \qquad (27)$$

$$\text{where} \quad \xi = \int \rho_b \mu_\delta u_\delta \, dy \qquad (21)$$

If we set u_δ = constant, in the integral ,

$$\xi = \frac{\mu_b u_\delta}{R T_b} \int p\,dx \qquad (28)$$

assuming a perfect gas.

Thus

$$\frac{d\xi}{dx} = \frac{\mu_b u_\delta}{R T_b} p \qquad (29)$$

or then

$$\frac{\xi}{d\xi}\frac{du_\delta}{u_\delta} = \frac{\int p\,dx}{p\,u_\delta}\frac{du_\delta}{dx} \qquad (30)$$

Since we are assuming an isentropic external flow,

$$\rho_\delta u_\delta \frac{du_\delta}{dx} = -\frac{dp}{dx} \qquad (31)$$

$$\therefore \quad \frac{\xi}{d\xi}\frac{du_\delta}{u_\delta} = -\frac{\int p\,dx}{p_\delta u_\delta^2 \rho_\delta}\frac{dp}{dx} \qquad (32)$$

or

$$\frac{2\xi}{d\xi}\frac{du_\delta}{u_\delta} = -\frac{\int p\,dx}{p^2}\frac{dp}{dx}\frac{R T_\delta}{\frac{u_\delta^2}{2}}$$

for perfect gas

$$= -\frac{\int p\,dx}{p^2}\frac{dp}{dx}\frac{\gamma-1}{\gamma}\frac{h_\delta}{\frac{u_\delta^2}{2}}$$

$$= -\frac{\int p\,dx}{p^2}\frac{dp}{dx}\frac{\gamma-1}{\gamma}\frac{h_\delta}{H_\delta} \qquad (33)$$

for $M_\delta^2 \gg 1$

Rewriting $\left(\rho_\delta/\rho - f_\eta^2\right)$,

$$\frac{\rho_\delta}{\rho} = \frac{h}{h_\delta} = \frac{T}{T_\delta} \qquad \text{for perfect gas}$$

$$\therefore \quad \left(\frac{\rho_\delta}{\rho} - f_\eta^2\right) = \left(\frac{H - \frac{u_\delta^2}{2} f_\eta^2}{h_\delta} - f_\eta^2\right) \tag{34}$$

$$= \frac{H - H_\delta f_\eta^2}{h_\delta} = \frac{H_\delta}{h_\delta}\left(g - f_\eta^2\right) \tag{35}$$

The momentum equation may then be written

$$N f_{\eta\eta\eta} + f f_{\eta\eta} - \frac{\int \rho\, dy}{\rho^2} \frac{d\rho}{dx} \frac{\gamma - 1}{\gamma}\left(g - f_\eta^2\right) = 2\xi\left(f_\eta f_{\eta\xi} - f_\xi f_{\eta\eta}\right) \tag{36}$$

The slip boundary conditions may be written, (to first order in λ) as

$$g(0) = g_b + b\lambda_b\left(\frac{\partial g}{\partial y}\right)_{y=0} \tag{37}$$

or

$$g(0) = g_b + \frac{b\lambda_b \rho_b u_\delta}{\sqrt{2\xi}}\, g_\eta\Big|_{\eta=0}$$

Temperature Jump

$$u(0) = a\lambda_b\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

or

$$u_\delta f_\eta\Big|_{\eta=0} = \frac{a\lambda_b \rho_b u_\delta^2}{\sqrt{2\xi}}\, f_{\eta\eta}\Big|_{\eta=0} \tag{38}$$

Velocity Slip

The form of the slip boundary conditions suggests that the appropriate expansion parameter for a perturbation solution about the zero order strong interaction solution would be $\varepsilon = \frac{\lambda_b \rho_b u_\delta}{\sqrt{2\xi_0}}$, where $\xi_0 = \frac{\mu_b u_\delta}{R T} \int p_b \, dx$. Evaluation of this small parameter in terms of M and Re shows that $\varepsilon = \sigma x^{-1/4}$, where $\sigma = \left(\frac{M\sqrt{C}}{\sqrt{Re}}\right)^{1/2} \sqrt{\frac{T_w}{T_0}} \, [.22]$. (See Appendix A.)

The expansion is applied directly to the transformed equations —

$$f = f_0(\eta) + \varepsilon f_1(\eta) + \cdots \tag{39}$$

$$g = g_0(\eta) + \varepsilon g_1(\eta) + \cdots \tag{40}$$

$$\beta = \beta_0 [1 + \varepsilon \beta_1 + \cdots] \tag{41}$$

$$\xi = \xi_0 [1 + \varepsilon \xi_1 + \cdots] \tag{42}$$

At the edge of the boundary layer, as $\eta \to \infty$, the following terms are expanded

$$p/p_\infty = p_0 (1 + \varepsilon p_1 + \cdots) \tag{43}$$

$$u_\delta = u_{\delta 0} (1 + \varepsilon u_1 + \cdots) \tag{44}$$

$$h_\delta = h_{\delta 0} (1 + \varepsilon h_1 + \cdots) \tag{45}$$

where p_1, u_1, h_1 are constants

Because the external flow is adiabatic, the stagnation enthalpy remains constant at H_δ.

When these forms are placed in the transformed equations, the following sets of equations are obtained

<u>Zeroth order in ε</u>

$$N f_{0\eta\eta\eta} + f_0 f_{0\eta\eta} + \beta_0 (g_0 - f_{0\eta}^2) = 0 \tag{46}$$

$$N g_{0\eta\eta} + f_0 g_{0\eta} = 0 \tag{47}$$

At $\eta = 0$; $g_0(0) = g_b$, $f_0 = 0$, $f_{0\eta} = 0$ (48)

At $\eta \to \infty$; $g_0 \to 1$, $f_{0\eta} \to 1$ (49)

The solutions of the zero[th] order equations are tabulated in [18]

First Order in ε

$$N f_{1\eta\eta\eta} + f_0 f_{1\eta\eta} + f_1 f_{0\eta\eta} + \beta_0 (g_1 - 2 f_{0\eta} f_{1\eta}) + \tfrac{3}{2} \beta_0 \beta_1 (g_0 - f_{0\eta}^2) = - f_{0\eta} f_{1\eta} + f_1 f_{0\eta\eta} \tag{50}$$

and

$$N g_{1\eta\eta} + f_1 g_{0\eta} + f_0 g_{1\eta} = - f_{0\eta} g_1 + f_1 g_{0\eta} \tag{51}$$

Rewriting

$$N f_{1\eta\eta\eta} + f_0 f_{1\eta\eta} + f_{0\eta} f_{1\eta} - 2\beta f_{0\eta} f_{1\eta} = -\beta_0 (g_1) - \tfrac{3}{2} \beta \beta_1 (g_0 - f_{0\eta}^2) \tag{52}$$

and $N g_{1\eta\eta} + f_0 g_{1\eta} + f_{0\eta} g_1 = 0$ (53)

Where B. C. are

$$\left.\begin{aligned} f_1'(0) &= a f_0''(0) \\ f_1'(\infty) &= 0 \\ f_1(0) &= 0 \end{aligned}\right\} \tag{54}$$

$$\left.\begin{aligned} g_1(0) &= b g_0'(0) \\ g_1(\infty) &= 0 \end{aligned}\right\} \tag{55}$$

4.2 Solution

The energy equation may be integrated immediately

$$g_1 = A g_0'(\eta) + B g_0'(\eta) \int_0^\eta \frac{e^{-\int \frac{f_0}{N} d\eta}}{(g_0')^2} d\eta \qquad (56)$$

The perturbation velocity $f_1'(\eta)$ can be written

$$f_1'(\eta) = C f_0''(\eta) + D f_0''(\eta) \int_0^\eta \frac{e^{-\int_0^\eta \frac{f_0}{N} d\eta}}{(f_0'')^2} d\eta + V_1 f_0'' + V_2 f_0'' \int_0^\eta \frac{e^{-\int_0^\eta \frac{f_0}{N} d\eta}}{(f_0'')^2} d\eta \qquad (57)$$

where V_1 and V_2 are obtained by variation of parameters.

$$V_1 = \int_0^\eta \frac{\left(\beta_0 g_1 + \beta_0 \frac{3}{2} \beta_1 (g_0 - f_0'^2)\right) f_0''}{e^{-\int \frac{f_0}{N} d\eta}} \left[\int_0^\eta \frac{e^{-\int \frac{f_0}{N} d\eta}}{(f_0'')^2} d\eta \right] d\eta \qquad (58)$$

$$V_2 = -\int_0^\eta \frac{\left(\beta_0 g_1 + \beta_0 \frac{3}{2} \beta_1 (g_0 - f_0'^2)\right)}{e^{-\int \frac{f_0}{N} d\eta}} f_0'' \, d\eta \qquad (59)$$

The condition at $\eta = 0$ permits the constants A and C to be evaluated, since all the other terms vanish there.

However, the condition at ∞ is identically satisfied by the functions, irrespective of the value of the constants B and D. It is then necessary to introduce some other requirement on the solution in order to establish a physically sensible solution. If the requirement is made that the various thicknesses which characterize the boundary layer possess finite dimensions, it is possible to evaluate the constants B and D. (See Appendix C).

When these constants are evaluated, the solution to the first order energy equation becomes

$$g_1(\eta) = b\, g_0'(\eta) \qquad (60)$$

The solution to the first order momentum equation can be written

$$f_1'(\eta) = a f_0''(\eta) + f_0''(\eta) \int_0^{\eta} \frac{e^{-\int_0^{\eta} \frac{f_0}{N} d\eta}}{(f_0'')^2}\, d\eta \left\{ \int_{\eta}^{\infty} \frac{\left[\beta_0 b g_0' + \beta_0 \frac{2}{2} \beta_1 (g_0 - f_0'^2)\right] f_0''\, d\eta}{e^{-\int_0^{\eta} \frac{f_0}{N} d\eta}} \right\}$$

$$+ f_0''(\eta) \int_0^{\eta} \frac{\left[\beta_0 b g_0' + \beta_0 \frac{2}{2} \beta_1 (g_0 - f_0'^2)\right]}{e^{-\int_0^{\eta} \frac{f_0}{N} d\eta}} \left[f_0'' \int_0^{\eta} \frac{e^{-\int_0^{\eta} \frac{f_0}{N} d\eta}}{(f_0'')^2}\, d\eta \right] d\eta \qquad (61)$$

Since the functions f_0, f_0'', g_0' and g_0 have already been calculated [18], it would be possible to evaluate the velocity and energy perturbations as numerical functions of η.

In order to calculate p_1, it is necessary to first calculate the displacement thickness, and then utilize the tangent wedge formula.

If we neglect terms of $O\left[\frac{1}{M_\delta^2}\right]$, the displacement thickness can be written

$$\delta^* \simeq \delta = \frac{\sqrt{2\xi}}{\rho_\delta u_\delta} \frac{H_\delta}{h_\delta} \int_0^{\infty} (g - f_\eta^2)\, d\eta \qquad (62)$$

Expanding in the small parameter ε, using the perfect gas law, and the results that $\rho_1 = 2p_1$ and $u_1 = 0$, we may write

$$\delta^* = \frac{R\sqrt{2\xi_0}\; T_0}{\mu_0 p_0 u_\delta} \left\{ \delta_0 + \varepsilon \delta_1 \right\} \qquad (63)$$

where

$$\delta_0 = \int_0^\infty (g_0 - f_0'^2)\, d\eta \qquad (64)$$

$$\delta_1 = \int_0^\infty (g_1 - 2 f_0' f_1')\, d\eta \qquad (65)$$

The integral, δ_1 , will be in the form $\alpha + p_1 \beta$, since the perturbation velocity profile is given as a linear function of p_1. Since $\delta_0^* = \frac{R\sqrt{2\xi_e}\, T_0\, \delta_0}{p_0 \rho_0 u_\delta}$, δ^* may be written $\delta^* = \delta_0^* \left[1 + \varepsilon \frac{\delta_1}{\delta_0}\right]$

The use of the tangent wedge formula requires that the slope of the displacement thickness be evaluated

$$\frac{d\delta^*}{dx} = \frac{d\delta_0^*}{dx}\left[1 + \varepsilon \frac{\delta_1}{\delta_0}\right] + \frac{d\varepsilon}{dx}\, \delta_0^* \frac{\delta_1}{\delta_0} \qquad (66)$$

$$\frac{d\varepsilon}{dx} = -\frac{1}{4}\frac{\varepsilon}{x} \quad , \quad \frac{d\delta_0^*}{dx} = \frac{3}{4}\frac{\delta_0^*}{x} \qquad (67)$$

$$\therefore \quad \frac{d\delta^*}{dx} = \frac{d\delta_0^*}{dx}\left[1 + \varepsilon \left(1 - \frac{1}{3}\right)\frac{\delta_1}{\delta_0}\right] \qquad (68)$$

or

$$\frac{d\delta^*}{dx} = \frac{d\delta_0^*}{dx}\left[1 + \varepsilon \frac{2}{3}\frac{\delta_1}{\delta_0}\right] \qquad (69)$$

The tangent wedge formula, as given in reference [8] is

$$\frac{p}{p_\infty} = \frac{\gamma}{2}(\gamma+1)\left(\frac{d\delta^*}{dx}\right)^2 M_\infty^2 \qquad (70)$$

or

$$p/p_\infty = p_0\left[1 + \varepsilon p_1\right] = \frac{\gamma}{2}(\gamma+1)\left(\frac{d\delta_0}{dx}\right)^2\left(1 + \frac{4}{3}\varepsilon \frac{\delta_1}{\delta_0}\right) M_\infty^2 \qquad (71)$$

Since $$\tau_0 = \frac{\gamma}{2}(\gamma+1)M_\infty^2\left(\frac{d\delta_0}{dy}\right)^2 \qquad (72)$$

then $$p_1 = \frac{4}{3}\frac{\delta_1}{\delta_0} \qquad (73)$$

The calculation of p_1 for the general case is quite complicated, since the numerical evaluation of several integrals is required in order to evaluate f_1' and hence, $\frac{\delta_1}{\delta_0}$. Although the complete calculation is possible in principle, it is of questionable utility, since the expansion parameter is of order δ/x. However, in order to give an estimate of the magnitude of p_1, we shall consider the case of the adiabatic wall. In addition, we shall set $p_1 = 0$ whenever it appears in the velocity profile. This is equivalent to setting the non-homogeneous terms to zero in the differential equations. If we suggest, on intuitive grounds, that the primary effect of slip on the velocity profiles is not through the change in the external flow field (i.e., p) but rather directly through the change in the boundary conditions, we can write the approximate solution of the momentum equation as

$$f_1'(\eta) = a f_0''(\eta) \qquad (74)$$

Evaluating δ_1, since $g_1 = 0$, for the adiabatic wall

$$\delta_1 = -2\int f_1' f_0' d\eta \qquad (75)$$

or

$$\delta_1 = -2a\int_0^\infty f_0' f_0'' d\eta \qquad (76)$$

$$\therefore \quad \delta_1 = -a \qquad (77)$$

and $$p_1 = -\frac{4}{3}\frac{a}{\delta_0} \qquad (78)$$

From Oguchi's paper[11], δ_o is evaluated from a solution of the zeroth order momentum equation. Oguchi's notation uses I_o to represent our δ_o.

$$I_o = 1.310 \quad \text{for air} \qquad (79)$$

Therefore, the nondimensional coefficient for the pressure perturbation is $p_1 = -a$. For the usual case of completely diffuse surface interaction, this constant a may be replaced by 1, since $a = \frac{2-\sigma}{\sigma} \times 2 \times 0.499$.

As demonstrated in Appendix A, $\varepsilon = \left(\frac{M\sqrt{C}}{\sqrt{Re_x}} \cdot \frac{T_w}{T_0}\right)^{1/2} (.22)$.

Therefore, our estimate of the induced pressure for the adiabatic wall is

$$p/p_\infty = p_o \bar{\chi} \left[1 - .22\left(\frac{M\sqrt{C}}{\sqrt{Re_x}}\right)^{1/2}\right] \qquad (80)$$

or

$$p/p_\infty = .555\,\bar{\chi}\left[1 - \left(\frac{M\sqrt{C}}{\sqrt{Re_x}}\right)^{1/2} .22\right] \qquad (81)$$

This can be written

$$p/p_\infty = .555\,\bar{\chi}\left[1 - \left(\frac{\bar{\chi}}{M}\right)^{1/2} .22\right] \qquad (82)$$

or

since $$\frac{\delta_o^*}{x} = .73\,\frac{\bar{\chi}^{-1/2}}{M} \qquad (83)$$

$$\therefore \quad p/p_\infty = .555\,\bar{\chi}\left[1 - .3\,\frac{\delta_o^*}{x}\right] \qquad (84)$$

An examination of equation (61) suggests that this is a maximum estimate for the decrease in induced pressure from the zeroth order values.

4.3 Skin Friction and Heat Transfer

To boundary layer order, the shear stress at the surface is

$$\tau_{y=0} = \mu_{(y=0)} \frac{\partial u}{\partial y}\Big|_{y=0} \quad (85)$$

Although it has been assumed that $\frac{\rho\mu}{\rho_b\mu_b} = N$ in order to provide a better fit to the true viscosity-temperature behavior over the entire temperature range encountered in a hypersonic boundary layer, this approximation is used only in the differential equations, and is not used to evaluate shear and heat transfer at the wall.

In our transformed system, thus,

$$\tau = \mu\rho \frac{u_\delta^2}{\sqrt{2\xi}} f_{\eta\eta} \quad (86)$$

We approximate $\mu(y=0)$ by $\mu_b \frac{T(0)}{T_b}$ where we have assumed that a linear fit is possible over the small temperature range $T_b - T_{(0)}$.

$$\therefore \mu(y=0) = \mu_b[1+\varepsilon T_1] \; (87) \quad \text{where } T_{(0)} = T_b(1+\varepsilon T_1)$$

The density at $y = 0$ is given by the perfect gas law,

$$\rho_{y=0} = \frac{P_\infty P_0(1+\varepsilon p_1)}{R T_b(1+\varepsilon T_1)} \quad (88)$$

where
$$\left.\begin{array}{l} P = P_\infty P_0 (1+\varepsilon p_1) \\ T_{(0)} = T_b (1+\varepsilon T_1) \end{array}\right\}$$

$$\therefore \rho_{y=0} = \rho_b[1+\varepsilon(p_1 - T_1)] \quad (89)$$ $\sim \rho_b$ is the zero[th] order density at the wall .

The shear stress can then be expanded in the form

$$\tau + \varepsilon\tau_1 = \frac{\mu_b \rho_0 u_\delta^2}{\sqrt{2\xi_0}}[f_{0\eta\eta} + \varepsilon f_{1\eta\eta}] \quad (90)$$

since $\rho_1 = 2p_1$.

The skin friction coefficient, defined by

$$\bar{C}_F = \tau \Big/ \tfrac{1}{2}\rho_\infty U_\infty^2 \qquad (91)$$

is given by

$$\bar{C}_F = \bar{C}_{F_0}(1 + \varepsilon C_{F_1}) \qquad (92)$$

$$= \frac{2\mu_b \rho_b U_1^2}{\rho_\infty U_\infty^2 \sqrt{2\xi_0}} f_{0\eta\eta}\left[1 + \varepsilon \frac{f_{1\eta\eta}}{f_{0\eta\eta}}\right] \qquad (93)$$

$$= C_{F_0}\sqrt{\frac{C}{Re_x}}\,\bar{\chi}^{1/2}\left[1 + \varepsilon \frac{f_{1\eta\eta}}{f_{0\eta\eta}}\right] \qquad (94)$$

where

$$\bar{C}_{F_0} = C_{F_0}\sqrt{\frac{C}{Re_x}}\,\bar{\chi}^{-1/2} \qquad (95)$$

Evaluating $f_{1\eta\eta}$, we obtain

$$f_{1\eta\eta}\Big|_0 = a f_{0\eta\eta\eta}\Big|_{\eta=0} + \frac{\gamma-1}{\gamma}\left[\int_0^\infty \frac{b g_0' + \tfrac{3}{2}p_1(g_0 - f_0'^2) f_0''}{e^{-b' \frac{f_0}{\pi}}}\,d\eta\right] \times \frac{1}{f_{0\eta\eta}(0)} \qquad (96)$$

If we attempt to estimate the magnitude of this slip correction to the skin friction on the same basis that we estimated the correction to the induced pressures, we can approximate the result for $f_{1\eta\eta}(0)$ by

$$f_{1\eta\eta}\Big|_0 = -\frac{\gamma-1}{\gamma} = -.286 \qquad (97)$$

$$\therefore\ \bar{C}_{F_1} = .568\,\bar{\chi}^{1/2}\cdot\sqrt{\frac{C}{Re_x}}\left[1 - .245\varepsilon\right] \qquad (98)$$

or since $\quad \varepsilon = .22\,\bar{\chi}^{-1/2}/M$

$$\bar{C}_F = .568\,\bar{\chi}^{1/2}\sqrt{\frac{C}{Re_x}}\left[1 - .05\frac{\bar{\chi}^{1/2}}{M}\right]\Big/\sqrt{\frac{C}{\sqrt{Re_x}}} \qquad (99)$$

For this case of the adiabatic wall, it is possible to make an estimate of the magnitude of the second term in the expression for $f_{1\eta\eta}$ by approximating f_o'' by its Blasius value, $f_o''(0)\, e^{-\int \frac{f_o}{\nu} d\eta}$.

Then, we may write

$$f_{1\eta\eta}\Big|_{\eta=0} = -\frac{\gamma+1}{\gamma}\left[1 - \tfrac{3}{2}\, p_1\, \delta_o\right] \quad (100)$$

where

$$\delta_o = \int_0^\infty \left(g_o - f_o'\right) d\eta \quad (64)$$

$$\delta_o \approx 1.310 \text{ for air}$$

If we approximate p_1 by -1, as suggested earlier, the result is then

$$f_{1\eta\eta}\Big|_0 \approx -.286\left\{2.97\right\} \quad (101)$$

or $f_{1\eta\eta}\Big|_0 \approx -.85$

or $$C_F = .568\bar{\chi}^{-1/2}\left\{1 - .18\frac{\bar{\chi}^{-1/2}}{M}\right\} \quad (102)$$

This also suggests that it may be possible to predict C_F from measured values of p_1, which are obtained from the measurements of induced pressures on adiabatic plates. If the above procedure represents the phenomenon adequately, then the deviations from the strong interaction pressures could be used in a formula of the form

$$C_{F_{SLIP}} = C_{F_o}\left\{1 - \varepsilon\, .286\left(1 - 1.97\, p_1\right)\right\} \quad (103)$$

where p_1 could be estimated from available data.

The heat transfer to the wall may be estimated more readily, but it should be noted that the energy transferred to a wall in slip flow is given by

$$q + u_1 \tau_w \qquad (104)$$

where q is the Fourier heat conduction term

u_1 is the slip velocity

τ_w is the shear at the wall.

In terms of our expansion parameter, ε ,

$$q = -k\frac{\partial T}{\partial y}\Big|_{y=0} = -\frac{\rho_0 \mu_0 u_s}{Pr\sqrt{2\xi_0}}\left[g_0'(0) + \varepsilon(g_1' - 2f_{0\eta\eta} f_{1\eta})\right]H_s . \qquad (105)$$

Since $$u_1 \tau_w = \varepsilon \frac{f_{1\eta} f_{0\eta\eta} \rho_0 \mu_0 u_s^3}{\sqrt{2\xi_0}} \qquad (106)$$

The total energy transfer to the wall is then

$$q + u_1 \tau_w = -\frac{\rho_0 \mu_0 u_s H_s}{Pr\sqrt{2\xi_0}}\left\{g_0'(0) - \varepsilon(g_1' - 2f_{0\eta\eta} f_{1\eta}(1-Pr))\right\} \qquad (107)$$

For any wall temperature and Pr = 1, $g_1'\Big|_0 = 0$, since $g_1'\Big|_0 = \Big|\frac{d}{d\eta}\left(e^{-\frac{f_0 L}{N}\eta}\right) = \Big|_0 f_0 e^{-\frac{f_0 L}{N}\eta} = 0$ at $\eta = 0$.

Thus, for Prandtl No. = 1, the slip work term is exactly balanced by the change in the Fourier heat conduction to the wall, and there is no net change in the energy transfer to the wall.

5 SUMMARY AND CONCLUSIONS

The solution to the problem of the effects of slip boundary conditions in the strong interaction region has been presented. Since the appropriate expansion parameter is of $O(1/\chi)$, the effects of slip, as obtained from this solution are only one of several possible higher order effects.

In view of the complexity of the general solutions, crude estimates of the slip effects on induced pressures and skin friction were made only for the adiabatic wall. These are presented in Figures 1 and 2. In addition, it was found that for the case of arbitrary constant wall temperature, and Pr = 1, there is no change in the heat transfer from zeroth order strong interaction solutions.

In order to evaluate the correctness of the estimates for induced pressure and skin friction, it would be necessary to integrate several functions of the zeroth order quantities. It would be desirable to do so, since this would show whether or not the significant departures from the strong interaction theory in the induced pressure can be attributed to slip at the surface. If it is decided that slip phenomena are present, and are important, then the use of the complete solutions would aid in the prediction of skin friction from pressure data.

Additional work is obviously necessary on the general problem of the hypersonic leading edge, both from the viewpoint of kinetic theory, starting from the tip, and also from the viewpoint of a set of continuum conservation equations such as the Navier-Stokes equations.

The apparent inadequacy of the hypersonic boundary layer equations in predicting aerodynamic quantities suggests that higher order effects may be both present and significant.

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Fig. 1 ESTIMATED INDUCED PRESSURES FOR ADIABATIC WALL WITH FIRST ORDER SLIP BOUNDARY CONDITIONS

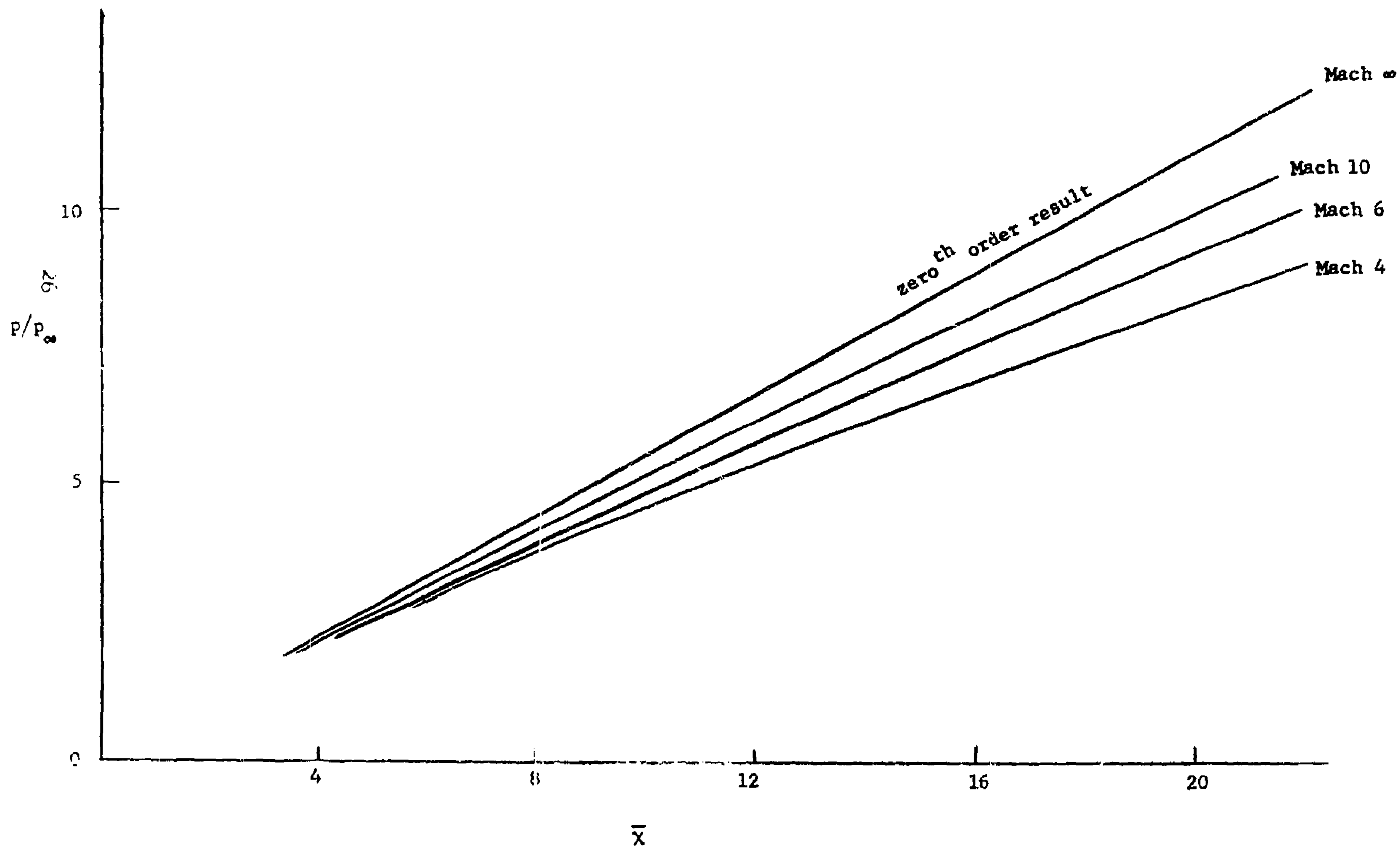
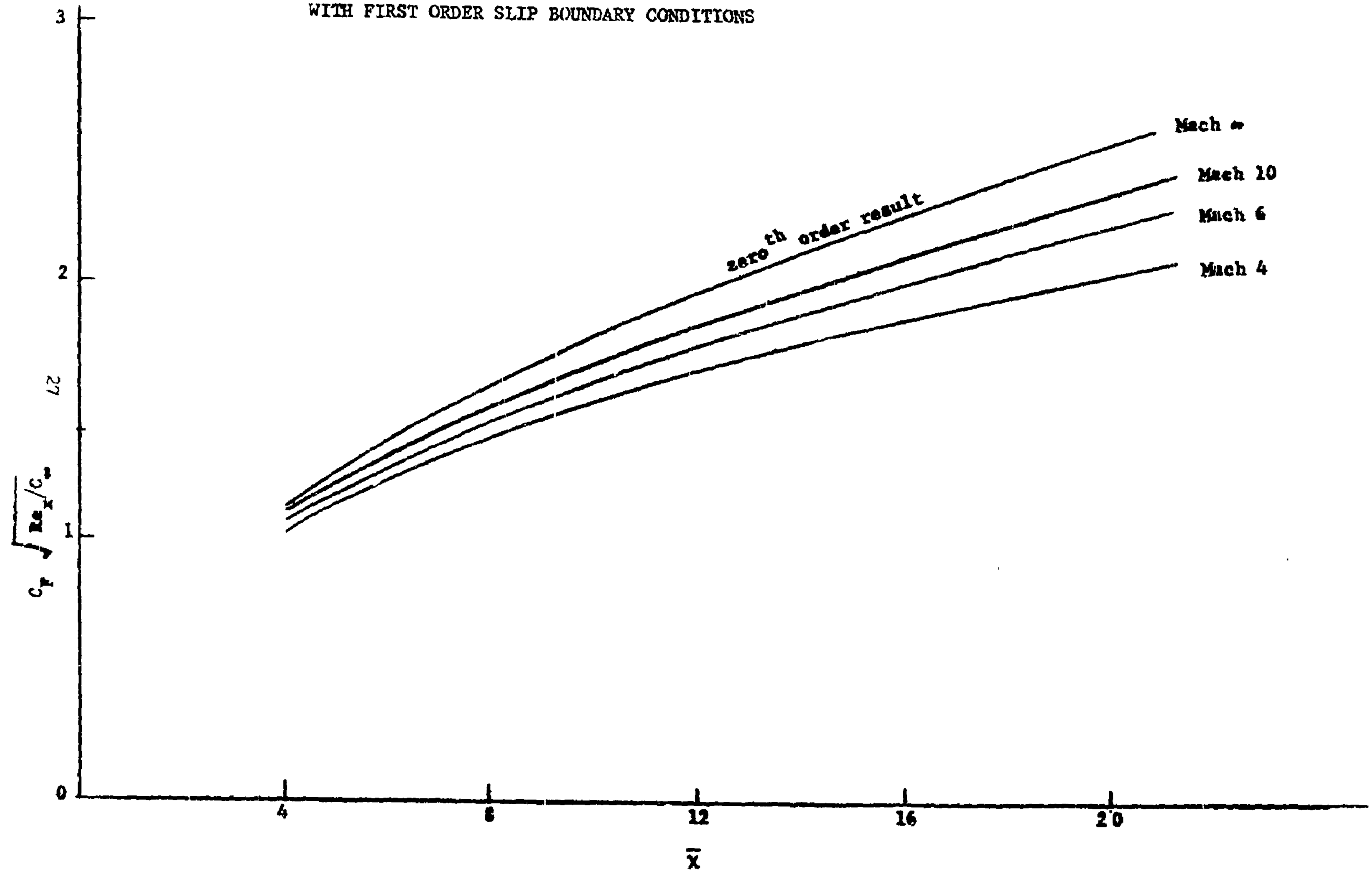


Fig. 2 ESTIMATED SKIN FRICTION COEFFICIENT FOR ADIABATIC WALL
WITH FIRST ORDER SLIP BOUNDARY CONDITIONS



APPENDIX A

EVALUATION OF SMALL PARAMETER

$$\varepsilon = \frac{\lambda_b \rho_b u_s}{\sqrt{2\xi_0}} \qquad \text{(A-1)}$$

$$\xi_0 = \int \mu_b \rho_b u_{s_0}\, dy = \frac{\mu_b u_s}{R T_b} \int p_b \rho_b\, dy \qquad \text{(A-2)}$$

$$= 2 \frac{\mu_b u_s}{R T_b} p_\infty \frac{M^3 \sqrt{C}}{\sqrt{Re_\infty}}\, x^{1/2}$$

$$\lambda_b = \lambda_\infty \frac{p_\infty}{p_b} \frac{\mu_b}{\mu_\infty} \sqrt{\frac{T_b}{T_\infty}} = \lambda_\infty \frac{\sqrt{Re}}{M^3 \sqrt{C}} \frac{\mu_b}{\mu_\infty} \sqrt{\frac{T_b}{T_\infty}} \sqrt{x} \qquad \text{(A-3)}$$

$$\rho_b = p_\infty \frac{M^3 \sqrt{C}}{\sqrt{Re}} \frac{1}{R T_b} \frac{1}{\sqrt{x}} \qquad \text{(A-4)}$$

$$\therefore\ \varepsilon = \sqrt{\frac{\gamma - 1}{8}} \left[\frac{T_b}{T_0}\right]^{1/2} \frac{\bar{\chi}^{1/2}}{M_\infty} \qquad \text{(A-5)}$$

For adiabatic flow, and Pr = 1

$$\varepsilon = .22 \left(\frac{M \sqrt{C}}{\sqrt{Re_y}}\right)^{1/2} . \qquad \text{(A-6)}$$

APPENDIX B

The expansion of the purely x-dependent quantities

$$p/p_\infty = p_0 \left[1 + \varepsilon p_1 + o(\varepsilon^2) + \cdots \right] \qquad \text{(B-1)}$$

$$\beta = \beta_0 \left[1 + \varepsilon \beta_1 + o(\varepsilon^2) + \cdots \right] \qquad \text{(B-2)}$$

where $\beta = 2 \dfrac{d \ln u_s}{d \ln \xi} \cdot \dfrac{h_s}{H_s}$ and $\beta_0 = \dfrac{\gamma - 1}{\gamma}$

$$u_s = u_{s0} \left[1 + \varepsilon u_1 + o(\varepsilon^2) + \cdots \right] \qquad \text{(B-3)}$$

$$\xi = \xi_0 \left[1 + \varepsilon \xi_1 + o(\varepsilon^2) + \cdots \right] \qquad \text{(B-4)}$$

where the coefficients , p_1, β_1, u_1, are constants

To first order in ε

$$\xi_1 = u_1 + 2 p_1 \qquad \text{from (21)} \qquad \text{(B-5)}$$

$$\beta_1 = 3/2\, p_1 \qquad \text{from (33)} \qquad \text{(B-6)}$$

$$u_1 = o\left[\frac{1}{M_d^2} \right] \qquad \text{from } H_s \equiv \text{constant} \qquad \text{(B-7)}$$

Therefore, to hypersonic order

$$\xi_1 = 2 p_1 \qquad \text{(B-8)}$$

$$\beta_1 = 3/2\, p_1 \qquad \text{(B-9)}$$

$$u_1 = 0 . \qquad \text{(B-10)}$$

APPENDIX C

The evaluation of the arbitrary constants in the general solution requires two additional boundary conditions, since the general solution identically satisfies the conditions that g_1 & $f_1' \to 0$ as $\eta \to \infty$. On physical grounds, we can require that the displacement and momentum thicknesses be finite, since otherwise our entire approach, being based strictly on the boundary layer hypothesis, would be invalid. This requirement is also in keeping with our assumption that the effects of first order slip can be expressed as a perturbation on the no slip solution. In addition, this requirement guarantees that the only singularity in the vertical velocity, and hence pressure, will occur at the leading edge.

The displacement thickness, in hypersonic flow, can be written as

$$\delta^* = \frac{\sqrt{2\xi}}{\rho_\delta u_\delta} \frac{H_\delta}{h_\delta} \int_0^\infty (g - f_\eta^2)\, d\eta \qquad (63)$$

neglecting terms of $\left(\frac{1}{M_\delta}\right)^2$

Expanding in ϵ yields $\delta^* = \{\delta_0^* + \epsilon\, \delta_1^*\}$ (C-1)

where $\delta_0^* = \left[\int_0^\infty (g_0 - f_0'^2)\, d\eta\right] \times$ function of x (C-2)

$\delta_1^* = \left[\int_0^\infty (g_1 - 2 f_0' f_1')\, d\eta\right] \times$ function of x (C-3)

The momentum thickness is

$$\Theta = \frac{\sqrt{2\xi}}{\rho_e u_e}\int_0^\infty f_\eta(1-f_\eta)\,d\eta \qquad \text{(C-4)}$$

which can be written similarly as

$$\Theta = \Theta_0 + \varepsilon\Theta_1$$

where

$$\Theta_0 = \left[\int_0^\infty f_{0\eta}(1-f_{0\eta})\,d\eta\right] \times \text{ function of x} \qquad \text{(C-5)}$$

$$\Theta_1 = \left[\int_0^\infty f_{1\eta}(1-2f_{0\eta})\,d\eta\right] \times \text{ function of x} \qquad \text{(C-6)}$$

$$+\left[\int_0^\infty f_{0\eta}(1-f_{0\eta})\,d\eta\right] \times \text{ function of x} \qquad \text{(C-7)}$$

The requirement of finite thickness at a station x thus requires that the integrals

$$\int_0^\infty (g_0 - f_0'^2)\,d\eta \ \text{(C-2)}, \quad \int_0^\infty (g_1 - 2f_0'f_1')\,d\eta \ \text{(C-3)}$$

$$\int_0^\infty f_{0\eta}(1-f_{0\eta})\,d\eta \ \text{(C-5)}, \quad \int_0^\infty f_1'(1-2f_{0\eta})\,d\eta \ \text{(C-6)}$$

all be bounded.

Since the results of the zeroth order solution are known to yield finite values for (C-2) and (C-5), we are then led to the condition that

$$\int_0^\infty (g_1 - 2f_0'f_1')\,d\eta$$

and

$$\int_0^\infty f_{1\eta}(1-2f_{0\eta})\,d\eta$$

be finite.

Thus
$$\left|\int_0^\infty (g_1 - 2f_0'f_1')\,d\eta\right| < \infty \qquad \text{(C-8)}$$

and
$$\left|\int_0^\infty f_1'(1-2f_0')\,d\eta\right| < \infty \qquad \text{(C-9)}$$

Since the asymptotic expansion for f_o' is (from reference [19])

$$f_o' = 1 + [\alpha_1(\eta-F)^{-(2\beta+1)} + \frac{\alpha_2}{2}(\eta-F)^{-1}]\exp[-\frac{(\eta-F)^2}{2}] + \ldots \tag{C-10}$$

we can write

$$\left| \int_0^\infty (g_1 - 2f_1'(1 + [\alpha_1(\eta-F)^{-(2\beta+1)} + \frac{\alpha_2}{2}(\eta-F)^{-1}]\exp(\frac{-(\eta-F)^2}{2}) + \ldots])\, d\eta \right| < \infty \tag{C-11}$$

and

$$\left| \int_0^\infty f_1'(1 - 2[\alpha_1(\eta-F)^{-(2\beta+1)} + \frac{\alpha_2}{2}(\eta-F)^{-1}]\exp[\frac{-(\eta-F)^2}{2}] + \ldots])\, d\eta \right| < \infty \tag{C-12}$$

The nature of the general solutions for f_1' reveals that the exponential terms in the integral will yield only a finite contribution. Therefore, we may write

$$\left| \int_0^\infty (g_1 - 2f_1')\, d\eta \right| < \infty \tag{C-13}$$

and

$$\left| \int_0^\infty f_1'\, d\eta \right| < \infty \tag{C-14}$$

This leads to the requirement that

$$\left| \int^\infty g_1\, d\eta \right| < \infty \tag{C-15}$$

The general solution for g_1 is

$$g_1 = A e^{-\int \frac{f_0}{N} d\eta} + B e^{-\int \frac{f_0}{N} d\eta} \int e^{\int \frac{f_0}{N} d\eta}\, d\eta \tag{C-16}$$

As $\eta \to \infty$, $f_0 \to \eta - F$, where F is a constant

$$\therefore \quad g(\eta\to\infty) = A e^{-\frac{(\eta-F)^2}{2}} + B e^{-\frac{(\eta-F)^2}{2}} \int_0^\eta e^{\frac{(\eta-F)^2}{2}}\, d\eta \tag{C-17}$$

or since $$e^{-\frac{(\eta-F)^2}{2}}\int_0^\eta e^{\frac{(\eta-F)^2}{2}}d\eta \rightarrow \frac{1}{\eta-F}$$

as $\eta \rightarrow \infty$,

$$\therefore \qquad g_{1(\eta\to\infty)} = \frac{B}{\eta - F} \quad + \text{exponential terms} \qquad \text{(C-18)}$$

The condition for finite $\int_0^\infty g_1 \, d\eta$ then requires that B = 0. Thus

$$g_1 = A e^{-\int^\eta \frac{F_0}{N} d\eta} \qquad \text{(C-19)}$$

where A is determined by the temperature jump condition at $\eta = 0$.

The requirement that $\int f_1' d\eta \equiv f_1$ be finite as $\eta \to \infty$ enables us to determine one arbitrary constant in the solution for f_1'. The general solution for f_1' is in the form

$$f_1' = f_0''(C + V_1(\eta)) + f_0'' \int \frac{e^{-\int \frac{f_0}{N} d\eta}}{(f_0'')^2} d\eta \left[D + V_2(\eta) \right] \quad (57)$$

where

$$V_1(\eta) = \frac{\gamma-1}{\gamma} \int_0^\eta \frac{[g_1 + \frac{3}{2} p_1 (g_0 - f_0'^2)]}{e^{-\int \frac{f_0}{N} d\eta}} \left[f_0'' \int_0^\eta \frac{e^{-\int \frac{f_0}{N} d\eta}}{(f_0'')^2} d\eta \right] d\eta \quad (58)$$

$$V_2(\eta) = -\frac{\gamma-1}{\gamma} \int_0^\eta \frac{[g_1 + \frac{3}{2} p_1 (g_0 - f_0'^2)]}{e^{-\int \frac{f_0}{N} d\eta}} f_0'' d\eta \quad (59)$$

C is determined by the slip condition at $\eta = 0$, and D will be determined by the behavior as $\eta \to \infty$.

Knowledge of the asymptotic behavior of the zero order solution is essential in order to evaluate D.

Writing

$$V_2(\eta) = V_{2\infty} - \int_\eta^\infty \frac{g_1 + \frac{3}{2} p_1 (g_0 - f_0'^2)}{e^{\int \frac{f_0}{N} d\eta}} f_0'' d\eta \quad \text{(C-20)}$$

since the integral $V_{2\infty}$ is finite.

For large values of η, the integral can be expanded asymptotically[19]

$$g_0 - f_0'^2 \approx \exp\left\{-\frac{(\eta - F)^2}{2}\right\} \left[2\alpha_1 (\eta - F)^{-2\beta+1} + \dots \right. \quad \text{(C-21)}$$

$$g_1 \approx E \exp\left[-\frac{(\eta-F)^2}{2}\right] \left[1 + \dots \right. \quad \text{(C-22)}$$

$$f_0'' \approx \exp\left(-\frac{(\eta-F)^2}{2}\right) \left[1 + \dots \right. \quad \text{(C-23)}$$

$$\int_\eta^\infty \sim \int_\eta^\infty \exp\left(-\frac{(\eta-F)^2}{2}\right) \left\{ E + 2\alpha_1 (\eta - F)^{-(2\beta+1)} + \dots \right\} d\eta \quad \text{(C-24)}$$

For the case where $E \neq 0$, heat transfer case

$$V_2(\eta)_{\eta\to\infty} = V_\infty - E\int_\eta^\infty D \exp\left[-\frac{(\eta-F)^2}{2}\right] + \cdots \, d\eta \tag{C-25}$$

For the case where E = 0, adiabatic wall

$$V_2(\eta)_{\eta\to\infty} \approx V_{\infty} - \int_\eta^\infty \alpha_1 \frac{\exp\left(-\frac{(\eta-F)^2}{2}\right)}{(\eta-F)^{2\beta+1}} \, d\eta + \cdots \tag{C-26}$$

Since the expansions for the integrals are only of exponential order, and contribute finite quantities,

$$V_2(\eta) = V_\infty - E \exp\left\{-\frac{(\eta-F)^2}{2}\right\} + \cdots \tag{C-27}$$

for $\eta \to \infty$

or

$$V_2(\eta) = V_\infty - 2\alpha_1 \exp\left(-\frac{(\eta-F)^2}{2}\right) + \cdots \tag{C-28}$$

Similarly for V_1, the expansion for large η can be written

$$V_1 \sim \int_0^\eta \frac{1}{\eta-F} \, d\eta + \cdots \tag{C-29}$$

or

$$V_1{}_{\eta\to\infty} \sim \ln(\eta-F) + \cdots \tag{C-30}$$

Considering the leading terms in the expansion for f_1,

$$\int_0^\infty f_1 \, d\eta \approx \int_0^\infty e^{-\frac{(\eta-F)^2}{2}}[1 + 0(\ln(\eta-F))] \, d\eta + \int_0^\infty \frac{1}{\eta-F}\left(D + V_{2\infty} - \alpha_1 \exp -\frac{(\eta-F)^2}{2} + \cdots\right) d\eta \tag{C-31}$$

The requirement that $\int_0^\infty f_1 \, d\eta$ be bounded is satisfied only be setting the arbitrary constant D equal to $-V_{2\infty}$.

$$\therefore \quad D = -V_{2\infty}$$

or

$$D = \frac{\gamma-1}{\gamma} \int_0^\infty \left\{\frac{g_1 + 3/2\, A(\beta \cdot f_0'^2)}{c^{\int f_0 d\eta}}\right\} f_0'' \, d\eta \, . \tag{C-32}$$

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